THE

MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS.

This Congress will be held at Cambridge, August 22nd-28th, 1912, Sir G. Darwin, F.R.S., being President. The Secretaries of the Congress are: Prof. E. W. Hobson, F.R.S., Christ's College, Cambridge, and Prof. A. E. H. Love, F.R.S., 34 St. Margaret's Road, Oxford.

Membership of the Congress is secured by the payment of £1, which entitles the subscriber to attend the Congress and to a copy of the Proceedings. The Treasurer is Sir J. Larmor, F.R.S., St. John's College, Cambridge. The Congress is divided into four sections, devoted respectively to Analysis, Geometry, Applied Mathematics, and to Philosophical, Historical and Educational questions.

In connection with the fourth section an International Commission on Mathematical Education has been appointed. At the opening meeting of the section the reports of this Commission will be presented by Prof. F. Klein, G.R.R. It is anticipated that some of the most distinguished mathematical teachers of Europe and America will be present.

The Mathematical Association has undertaken to organise an Exhibition in this section, the arrangements being in the hands of:

Mr. P. Abbott, -	Models, Apparatus, and Books.
Mr. F. C. Boon, -	Work of Secondary Schools.
Dr. T. P. Nunn and Miss Punnett,	Work of Elementary Schools and Training Colleges.
Mr. F. J. Whipple,	Mathematical Instruments and Calculating Machines.

It is very important that the exhibition should be representative and worthy of such an occasion. The Council of the Mathematical Association earnestly request the members of the Association and others interested to co operate for this purpose, and to give all the help and encouragement in their power.

The leaflet sent out with this number of the *Gazette* gives all necessary details as to the proposed exhibition. With respect to the subjects of discussion at the meetings of the section, the following will not be without interest to readers of the *Gazette*.

First Meeting.—Presentation of the reports of the national sub-commissions.

Second Meeting.—The rôles of intuition and experiment in the mathematical teaching of secondary schools.

Third Meeting.—Mathematics and Physics: the extent to which the University student in physics should carry his mathematical studies.

As a preparation for the discussions at the second and third meetings enquiries have been made from all available sources upon the following lines:

Second Meeting.—I. Measurement and Estimation of Magnitudes. In what types of schools, to what extent, at what ages, and in what classes do students

(*a*) pass to numerical calculations from geodesic measurements; use the theodolite, the chain, etc.?

(*b*) engage in astronomical observations and measurements, and study the problems connected therewith; use photographic and other general apparatus?

II. Drawing and Graphic Representation.

(*a*) study Descriptive Geometry (oblique projection, plane and elevation, central projection, theory of shadows)? Is there any co-ordination between this teaching and that of stereometry? Is the instruction given by the teacher of mathematics or by the drawing master?

(*b*) Graphical methods (representation of functions on millimetric paper, scalars and vectors, graphical calculation, and, in particular, graphic statics, evaluation of areas by squared paper or the planimeter)?

III. Numerical Calculations and Evaluations.

(*a*) Approximate calculations by aid of decimals?

(*b*) use of slide rule?

(*c*) use of tables for logarithms and trigonometrical functions? Use of tables of square and cube roots? and tables of mortality? Is the value of such tables exhibited by the aid of examples?

(*d*) Numerical and graphical solution of equations by approximation (Newton's rule, *Regula Falsi*, nomographic methods)?

Third Meeting.—Mathematics and the student taking up physics at the Universities:

(i) What branches of Mathematics should form part of the regular course of such students?

Should any distinction be drawn between the mathematical training of students whose courses will be in experimental rather than in theoretical physics?

Do teachers of mathematics keep clearly in mind the requirements of students in physics? Are there special mathematical courses for such students?

To what extent do the students of pure mathematics take courses in (*a*) mechanics; (*b*) other subjects which are now classed under "mathematical physics"?

(ii) To what extent are the modern graphical methods of integration and nomography cultivated in the Universities?

Are students in physics expected to take courses in descriptive geometry, numerical calculation, numerical solution of differential equations, and the method of least squares?

Are they trained in the use of the slide rule, calculating machine, and planimeter?

Are these instruments dealt with in the ordinary routine of physical work, or are there special courses or exercises for the purpose?

(iii) How are the mathematical exercises for students in physics organised? Are they part of the general laboratory routine?

How far do the Professor and his assistants enter into personal relations with the students?

(iv) What is your personal opinion as to the present organisation of the teaching of physics in this respect?

Have you suggestions to offer as to the extension or reduction of the amount of mathematical instruction for students in physics, as to the division of such students into appropriate groups, or as to the general organisation of their courses?

FOUR FOURS. SOME ARITHMETICAL PUZZLES.

An arithmetical amusement, said to have been first propounded in 1881, is the expression in the ordinary arithmetic and algebraic notation of the consecutive integers from 1 upwards, as far as practicable, by the use of four "4"s. I have mentioned the problem in my *Mathematical Recreations*, but my friend Mr. Oscar Eckenstein has now carried the solutions considerably further. I think a bare statement of our procedure may be of interest, and perhaps some readers of the *Gazette* may amuse themselves by filling in the details of the analysis or applying similar methods to higher numbers.

The solutions will vary according to what we mean by ordinary arithmetic and algebraic notation. Here I will assume that we allow the use of brackets and the symbols for square roots, decimals (simple and repeating), factorials, and subfactorials,* as well as those for addition, subtraction, multiplication, and division, and that we exclude indices (other than first powers) not expressible by a "4" or "4"s, and roots (other than square roots used a finite number of times). On this assumption we can express by four "4"s every number up to 873.

The following numbers, forming what I will call the *series* α, are expressible by *one* "4": 1, 2, 3, 4, 6, 9, 24, 265, 720, ..., and we may conveniently use them in this form instead of writing them in the more cumbrous "4" notation. Moreover, since 1 is expressible by one "4," if we can express a number by less than four "4"s we can, by multiplying by 1 a sufficient number of times, express it by four "4"s. From this series it follows that if m and n are two numbers such that $n-m$ is less than 10, then every number between m and n is expressible by m or n and one "4."

The numbers 1 to 13, 15 to 18, 20 to 28, 30, 33, 36, forming the *series* β, are expressible by *two* "4"s. Hence, if m and n are two numbers such that $n-m$ is less than 41 (and is not equal to 33), every number between m and n is expressible by m or n and two "4"s. If $n-m$ is equal to 33, such expressions for the numbers $m+14$ and $m+19$, are not obtainable in this way.

The following numbers, forming the *series* γ, are also expressible by *two* "4"s: 48, 54, 60, 64, 72, 80, 81, 96, 106, 120, 144, 180, 216, 240, 241, 256, 274, 288, 289, 320, 360, 455, 480, 504, 512, 530, 576.... Hence a number expressible as the algebraic sum or product of any two numbers in the series β and γ is expressible by four "4"s, as also is a number expressible in a form like $(\beta \pm \alpha)/\alpha$ or $(\gamma \pm \alpha)/\alpha$. We can thus obtain an expression by four "4"s for every number up to $576+28$ except for $576-14$, $576+19$, and a few numbers between $360+14$ and $455-14$.

* Subfactorial n is written $\underline{n|}$ or n¡ and is equal to

$$n!\,(1-1/1!+1/2!-1/3!+\ldots\pm 1/n!).$$

Now $562=2(265+4^2)$, and the expression by four "4"s of 595 and of the numbers between 374 and 441 is covered by the argument in the next paragraph.

The following numbers forming the *series* δ are expressible by *three* "4"s: 371, 372, 384, 385, 393, 396, 399, 400, 402, 405, 409, 411, 424, 431, 432, 441, 445, 450,... 540, 592, 594, 600, 603, 608, 614, 618, 625,.... This series combined with series α gives an expression by four "4"s for every number from $371-4$ to $450+4$, and for every number from $592-4$ to $625+4$, except $411+5$ and $411+8$, and these are obtainable by the method given in the last paragraph. We have now obtained expressions for all numbers up to 629.

The numbers in the series β and γ combined with series α give an expression by *three* "4"s for every number from 1 to 100 except 91. Now 720 is expressible by one "4." Hence all numbers from 620 to 820, except 720 ± 91, are expressible by four "4"s. Of the two numbers 720 ± 91, one, 629, has already been obtained, and the other, 811, $=3\times 265+4^2$. The numbers 822, 824, 826, 840, 841, 864, 867, are also expressible by *three* "4"s. These numbers, combined with the numbers in series α, enable us to express the numbers from 821 to 850, 855, 860 to 871, and 873 by four "4"s; and the other numbers between 850 and 873 are expressible directly. Hence we have now obtained expressions by four "4"s for all numbers up to 873.

If we note that 729, 795, 935, 1060, 1080, 1296,... 1854... are in series γ, and that 900, 901, 910, 927, 936, 954, 960, 961, 964, 976, 994, 1000, 1008, 1009, 1024, 1031, 1036, 1040, 1089, 1096, 1104, 1116, 1120, 1134, 1152, 1156, 1166, 1175, 1200, 1224, 1232, 1236, 1250, 1331, are in series δ, most of the numbers from 874 to 1335 can be obtained by similar operations.

In the above work I have not used numbers like 44 and 444, which directly introduce a scale of notation, though I think their use would be as legitimate as that of decimals. Also, though numbers like 1, 2, 3, 6,... are at once expressible by one "4," I regard numbers like 22 or $\cdot 2$ as inadmissible, since only the usual arithmetic notation is allowed.

Without the use of subfactorials it is comparatively easy to express every number, except 113 and 157, up to 162 by four "4"s, but as yet I have failed to find such expressions for 113 and 157, and in the expressions for 103 and 109 I have been driven to use the number 44.

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February, 1912.

THE THEORY OF ORDER, AS DEFINED BY BOUNDARIES.

IV. (*continued*).

Note. *Alternative proof of Theorem* 22.

This theorem follows as a corollary of the proposition that: *If in any system of conjugate pairs we collate one conjugate pair identically with itself (or with itself reflected), we cannot collate any other conjugate pair except identically, or with the pair into which it reverts with respect to the first pair.* For consider the scheme

$$\begin{array}{lcccccc}
(1) & a & a' & b & b' & c & c' \\
(2) & a & a' & c & c' & (b) & (b') \\
(3) & a' & a & b' & b & c' & c.
\end{array}$$

In (1)-(2) we represent a collation which collates (aa') identically, and (bb') with (cc'). The collation (1)-(3) is the one which determines the system of conjugate pairs. And we see that the collation (2)-(3) is reciprocal, and

we can therefore write in b and b' in line (2), over c' and c in line (3), and therefore under c and c' in line (1). We now see that (1)-(2) is a reversion, with respect to (aa'). If (aa') is reflected in line (2), the above argument will stand, if we merely interchange lines (1) and (3).

V. The Addition Theorem.

It may be that some of my readers who have followed me so far will be inclined to say, "What a lot of fuss to make about a conclusion so obvious as that the problems of geometry may logically be regarded as problems in the theory of Order!" If so, I quite sympathise with them; indeed, I acted on this view a good many years ago, when I maintained, as I still maintain, that the general considerations which I then advanced were logically sufficient to prove it. But not only the adequacy of my proof, but the fact itself, was strenuously denied, at the time, and it may be therefore that some one will say now, "There must be some flaw in your argument somewhere, even though I cannot lay my finger on it for the moment. You must have slipped in some unacknowledged premises somewhere." Very probably he would be having Hilbert's reasoning at the back of his mind in saying so. If so, I would remind him that Hilbert only proved that *his* Axioms of Order were inadequate, not that mine are so. In any case his division of the subject is different from mine, since what he calls Axioms of Order pre-suppose his Axioms of Connection, including the definition of a "straight line," though in effect he subsequently admits that without his Axioms of Congruence his definition is inadequate. But in order to remove remaining doubts, I propose to show how, instead of the alphabetical catalogue we have hitherto been employing, we may use a numerical one, by means of which, after assigning numbers in a quasi-arbitrary fashion to the units of a group, we can actually carry out unique collations among them, so that the possibility of unique collation is no longer a hypothesis, but a demonstrable fact.

In order to assign numbers as names in this way, to the units of a group of the first order, it is, of course, necessary that any number should denote one unit uniquely. But there is no reason why various numbers should not denote the same unit. If we assign, in the first instance, numbers, whole or fractional, from a limited part of the number series, both the limiting numbers must be assigned to one unit, as otherwise it would have only one unit in the number group contiguous to it. It will be seen presently that every unit will have not only two, but any number of numbers, besides the one initially chosen for it. For the sake of generality, I shall not use actual numbers in the discussion, but represent them by small letters of the alphabet; which, as when I used them to represent catalogue names before, are not to be held to imply any alphabetical order. Thus we may choose our numbers initially from, say, a to $(a+t)$, inclusive; a and $(a+t)$ denoting one and the same unit.

It may now be shown that transformations with respect to any given system of conjugate pairs form an assemblage of what may be regarded as absolute continuous magnitudes, the rule of combination being addition. (See *Trans. Amer. Math. Soc.*, Vol. iii., No. 2, p. 267.) That is, if we may represent a transformation which changes a into x by $(x-a)$, the numbers assigned to the units of the group may be regarded as the transformations which would change the name of the unit called zero to these names. A transformation t therefore changes a, and also every other unit, into itself again. Therefore $(x+nt)$, where n is any positive or negative integer, means the same unit as x. And one which changes an unit into its conjugate will be represented by $\frac{1}{2}t$, one which changes

it into an unit of the other conjugate pair of its quartette by $\frac{1}{4}t$, and so on. If, with respect to the conjugate pair containing b, a reverts into c, then the transformation which changes a into b also changes b into c, and therefore $(b-a)$ is equal to $(c-b)$, or

$$b=\tfrac{1}{2}(a+c) \quad\text{...(i)}$$

is the equation indicating reversion. The conjugate pairs of the positive system may be catalogued by the same numbers, but here it must be remembered that x and $(x+\frac{1}{2}t)$ represent the same conjugate pair, though it is reflected.

It will be noted that by naming the units of the group in the way described we *ipso facto* establish a positive system of conjugate pairs among them. It must not, however, be supposed that this imposes any fresh restriction upon our freedom of choice in cataloguing the group. Nor does the quasi-arbitrary exchange of catalogue names for numbers introduce any fresh arbitrariness into the catalogue, since no fresh assumption has been made as to the mere order of the units, while the old alphabetical catalogue told us nothing about them except their order. The numerical catalogue thus arrived at I shall call the *primary* catalogue; any catalogue which can be derived from it by unique collation or by transformation with respect to any system of conjugate pairs (themselves determined by any inversion), I shall call a *derived* catalogue. The system of conjugate pairs established by the act of assigning the numbers I shall call the *primary* system.

The problem of unique collation may now be formulated thus: We wish to change the names assigned to the units of a group so that any three, A, B, and C, say, may be given any three new names, selected at random, but, having done that, the new name of any fourth unit, D, is to be uniquely determined. That is to say, with reference to the three changes, a into a', b into b', and c into c', we must be able arithmetically to determine any d' from the corresponding d. The new number d' will, in mathematical language, be some function of d, and of the changes in the other numbers. I have to show that this function is mathematically defined by the assumptions on which my theory of Order is based, and the theorems which I have deduced from them.

It is convenient not to change directly from the old to the new catalogue, but, for the sake of symmetry, to change both the old and the new catalogues into a third, which I shall call the *intermediate* catalogue, by equating functions of the old with analogous functions of the new numbers. These functions I call the *projective functions*, and it will be seen that upon their form will depend the primary system of conjugate pairs, the system, that is, with respect to which transformation is addition, of each catalogue.

The projective functions in the two cases need not therefore always be the same, but both must be one-valued functions, so that to each value of the argument one, and only one, number in each catalogue shall correspond; and moreover the two projective functions which we equate together must respectively be functions also of the numbers denoting the units which determine the collation, such that they become identically equal if the arguments are put respectively equal to the numbers denoting a pair whose collation was given. Expressions of the form

$$\frac{f(x-c)-f(a-c)}{f(b-c)-f(a-c)},$$

where x is the argument, and a, b, c denote the units whose collations are given, fulfil these conditions, provided that $f(x)$ is a function which (i) is one valued for all values of x; and (ii) is infinite for zero value of the argument. To determine d', the new number collated with any old

number d, we equate the above expression, with d in place of x, with a similar expression in which every letter bears an accent; including even f, unless the same primary system of conjugate pairs is to be employed in the new as in the old catalogue; in which case f' would be the same as f. We see at once that the equation is satisfied if x is a and x' is a', for both expressions vanish, whatever the other letters may be. So also if x is b and x' is b', both sides reduce identically to unity; and if x is c and x' is c', both become infinite.

We may now further determine the nature of the function $f(x)$ by expressing numerically some of the theorems of Order which we have deduced above. First, let us express a reversion, say that B reverts into D, with respect to a boundary (AC). That is, when we collate A, B, C with A, D, C, respectively, we find that D is collated with B. Equating the two projective functions,

$$\frac{f(d-c)-f(a-c)}{f(b-c)-f(a-c)}=\frac{f(b-c)-f(a-c)}{f(d-c)-f(a-c)},$$

or,

$$\{f(d-c)-f(a-c)\}^2=\{f(b-c)-f(a-c)\}^2,$$

and therefore, either,

$$f(d-c)=f(b-c),$$

or else,

$$f(d-c)+f(b-c)=2f(a-c).$$

Since D is not the same unit as B, it is determined by the latter equation.

Now suppose that (AC), the boundary of our reversion, is a conjugate pair in the primary system; then for $(a-c)$ we may write $\frac{1}{2}t$. And, if we write y for $(d-c)$, we may write $-y$ for $(c-d)$, which is equal to $(b-c)$. Hence, whatever be the value of y, we have, identically,

$$f(y)+f(-y)\equiv 2f(\tfrac{1}{2}t). \quad\text{.................................(ii)}$$

Secondly, let us express an inversion, such for example as collates A, B, C, D respectively with D, C, B, A. Since this collation is possible whatever units the letters may represent, we have again identically

$$\frac{f(d-c)-f(a-c)}{f(b-c)-f(a-c)}\equiv\frac{f(a-b)-f(d-b)}{f(c-b)-f(d-b)},$$

whence

$$f(d-c).f(c-b)-f(d-b).f(d-c)+f(b-c).f(d-b)$$
$$\equiv f(a-c).f(c-b)-f(a-b).f(a-c)+f(b-c).f(a-b),$$

in which a function of d is identically equated to the same function of a. Since d and a are independent quantities, the functions on both sides of the equation must be independent of both of them. But by means of the formula (ii) the expression on the right of the equation can be put in the form

$$f(a-b).f(c-a)+f(b-c).f(a-b)+f(c-a).f(b-c)$$
$$-2\{f(a-b)+f(b-c)+f(c-a)\}f(\tfrac{1}{2}t)+\{f(\tfrac{1}{2}t)\}^2,$$

which shows that it is symmetrical in respect of a, b and c, and must therefore be independent of them all, or none. It is therefore an absolute constant; say k. If we now put x for $(a-b)$ and y for $(b-c)$, and therefore $(x+y)$ for $(a-c)$, we may determine k by putting $x\equiv y\equiv\frac{1}{4}t$.

Substituting, and rearranging the terms, we get

$$\{f(x+y)-f(\tfrac{1}{2}t)\}\{f(x)-f(\tfrac{1}{2}t)+f(y)-f(\tfrac{1}{2}t)\}$$
$$=\{f(x)-f(\tfrac{1}{2}t)\}\{f(y)-f(\tfrac{1}{2}t)\}-\{f(\tfrac{1}{4}t)-f(\tfrac{1}{2}t)\}^2.$$

Now $f(\frac{1}{4}t)$ and $f(\frac{1}{2}t)$ cannot be equal, for they denote two units belonging to different conjugate pairs in a quartette. Hence we may divide by their

difference squared, and if we define a new function, which is also one-valued, and infinite for zero value of the argument, by

$$F(x) \equiv \frac{f(x)-f(\tfrac{1}{2}t)}{f(\tfrac{1}{4}t)-f(\tfrac{1}{2}t)},$$

we get $$F(x+y)\{F(x)+F(y)\} \equiv F(x).F(y)-1.$$

In the same way equation (ii) becomes

$$F(x) \equiv -F(-x), \quad \text{.................................(iii)}$$

from which we see that $F(x)+F(y)$ does not vanish unless y is equal to $(-x)$ increased or diminished by an integral multiple of t, in which case, as we already know, $F(x+y)$ is infinite.

We may, therefore, without ambiguity, divide both sides of the above equation by $F(x)+F(y)$, and obtain

$$F(x+y) \equiv \frac{F(x).F(y)-1}{F(x)+F(y)}. \quad \text{..............................(iv)}$$

This is the addition equation which, as is well known, suffices to determine F as one of the functions of single periodicity; $F(x)$ being either $\cot(x)$, $\coth(x)$, or $\frac{F}{x}$ (omitting arbitrary constants), according as the period is finite and real, finite and imaginary, or infinite.

We are therefore able not only to actually carry out any unique collation in our primary catalogue, but to derive from it new catalogues in which the primary systems of conjugate pairs shall be determined by hyperbolic or by reciprocal projective functions. It is obvious that these correspond to the geometries of Riemann, Lobatchewsky, and Euclid. The only other point to which I wish to draw attention now is this. In the various systems of Metageometry we are supposed to be dealing with various Spaces. But when employing different projective functions in a numerical catalogue, and so deriving fresh catalogues, we do not conceive ourselves as dealing with different groups of units, or different kinds of Order. We recognise that the differences are differences of Names, and not of things.

EDWARD T. DIXON.

The Hard, Hythe,
Southampton, Feb. 13, 1911.

ERRATA.

p. 175, line 21 from top, *for* Q in line (1), *read* L in line (1).
,, line 18 from bottom, *for* F_1 with L_2: consequently also L_1 with F_2, *read* F_2 with L_3: consequently also L_2 with F_3.
p. 178, line 18 from bottom, *for* (1)-(3), *read* (1)-(2).
,, line 17 from bottom, *for* (1)-(2), *read* (1)-(3).

SUGGESTED NOTATION FOR RATIOS AND CROSS-RATIOS.

I WISH to call attention to the value, for some purposes, of the notation

$$a\left(\frac{b}{c}\right) \text{ for the ratio } \frac{ab}{ac}; \quad \text{and} \quad \frac{a}{b}\left(\frac{c}{d}\right) \text{ for the cross-ratio } \frac{ac}{ad}:\frac{bc}{bd}.$$

For instance: in Menelaus' theorem for the property of a transversal meeting the sides of a triangle ABC in the points P, Q, R, the first mentioned notation makes the property shine out very clearly. The equation in this form is

$$P\left(\frac{B}{C}\right).Q\left(\frac{C}{A}\right).R\left(\frac{A}{B}\right)=1,$$

which conspicuously separates the points on the transversal from the angular points of the triangle.

Thus, in Durell's *Plane Geometry*, vol. i., p. 119, where A, D, F are collinear points on the sides of a triangle BGC, and L, M, N are points (to be proved collinear) on the sides of a triangle QRP, the equations he obtains may be written

$$L\left(\frac{R}{Q}\right)=A\left(\frac{B}{G}\right),$$

$$M\left(\frac{P}{R}\right)=D\left(\frac{G}{C}\right),$$

$$N\left(\frac{Q}{P}\right)=F\left(\frac{C}{B}\right).$$

The application of Menelaus' is wonderfully obvious, the transversals standing out boldly from the triangular points.

The cross-ratio expression $\frac{a}{b}\left(\frac{c}{d}\right)$ for four points on a range is a fairly obvious extension of the notation for a single ratio, being a condensation of

$$a\left(\frac{c}{d}\right)\div b\left(\frac{c}{d}\right).$$

The first umbral fraction $\frac{a}{b}$ may be called the operator, and $\left(\frac{c}{d}\right)$ the operand. The two are interchangeable, *i.e.* $\frac{a}{b}\left(\frac{c}{d}\right)=\frac{c}{d}\left(\frac{a}{b}\right)$. This is its simplest property.

Particular cases are $\frac{a}{b}\left(\frac{m}{m}\right)=1$; and $\frac{a}{\infty}\left(\frac{c}{d}\right)=a\left(\frac{c}{d}\right)$.

To invert $\frac{a}{b}\left(\frac{c}{d}\right)$ we must invert either the operator or the operand, it does not matter which. Thus, if the given cross-ratio $=\lambda$, $\frac{a}{b}\left(\frac{d}{c}\right)=\frac{b}{a}\left(\frac{c}{d}\right)=\frac{1}{\lambda}$.

If two cross-ratios with a common operator (or common operand) are to be multiplied, we may write the result as follows:

$$\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{a}{b}\left(\frac{x}{y}\right)=\frac{a}{b}\left(\frac{c}{d}\cdot\frac{x}{y}\right)=\frac{a}{b}\left(\frac{x}{d}\cdot\frac{c}{y}\right)=\frac{a}{b}\left(\frac{x}{y}\cdot\frac{c}{d}\right),$$

as is obvious on expansion. Hence the parts of a compound operand are commutative.

The fact that $\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{a}{b}\left(\frac{d}{c}\right)=1$, as stated above, is a particular instance of this theorem.

Another useful application of the theorem is

$$\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{a}{b}\left(\frac{d}{e}\right)=\frac{a}{b}\left(\frac{c}{d}\cdot\frac{d}{e}\right)=\frac{a}{b}\left(\frac{c}{e}\right);$$

and similarly $\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{b}{e}\left(\frac{c}{d}\right)=\frac{a}{e}\left(\frac{c}{d}\right)$, in which the operators are now compounded and simplified.

Euler's theorem $ab\,.\,cd+ac\,.\,db+ad\,.\,bc=0$ can be put into the form

$$\frac{ab\,.\,cd}{ad\,.\,cb}+\frac{ac\,.\,bd}{ad\,.\,bc}=1,$$

$$\textit{i.e.}\quad \frac{a}{c}\left(\frac{b}{d}\right)+\frac{a}{b}\left(\frac{c}{d}\right)=1;$$

so that if we interchange, cross-wise, a pair of letters between operator and operand, we obtain the complementary cross-ratio.

Hence the two fundamental equations for one set of cross-ratios are

(1) $\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{a}{b}\left(\frac{d}{c}\right)=1$, the reciprocal equation ;

(2) $\frac{a}{b}\left(\frac{c}{d}\right)+\frac{a}{c}\left(\frac{b}{d}\right)=1$, the complementary equation.

To evaluate such a ratio as $\frac{a}{d}\left(\frac{c}{b}\right)$.

We have
$$\frac{a}{d}\left(\frac{c}{b}\right)=\frac{1}{\frac{a}{d}\left(\frac{b}{c}\right)}=\frac{1}{1-\frac{a}{b}\left(\frac{d}{c}\right)}=\frac{\frac{a}{b}\left(\frac{c}{d}\right)}{\frac{a}{b}\left(\frac{c}{d}\right)-1}=\frac{\lambda}{\lambda-1},$$
by the use of equations (1) and (2).

The interesting fact that
$$\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{a}{c}\left(\frac{d}{b}\right)\times\frac{a}{d}\left(\frac{b}{c}\right)=-1$$
is no doubt most easily *proved* by putting the cross-ratios in their expanded form, but the above form is an expressive way of representing the identity.

It is also easy to prove it from equations (1) and (2), thus :
$$\text{The product}=\frac{\frac{a}{b}\left(\frac{c}{d}\right)}{\frac{a}{c}\left(\frac{b}{d}\right)}\left\{1-\frac{a}{b}\left(\frac{d}{c}\right)\right\}=\frac{\frac{a}{b}\left(\frac{c}{d}\right)-1}{\frac{a}{c}\left(\frac{b}{d}\right)}=-1.$$

Or we may proceed thus :
$$a\left(\frac{c}{d}\right)\times a\left(\frac{d}{b}\right)\times a\left(\frac{b}{c}\right)=1, \text{ obvious on expansion,}$$
and we have merely to show that
$$b\left(\frac{c}{d}\right)\times c\left(\frac{d}{b}\right)\times d\left(\frac{b}{c}\right)=-1, \text{ which is similarly obvious.}$$

Note. The notation $\frac{a}{b}\left(\frac{c}{d}\right)$ is equally handy if the letters denote, instead of mere points on the range, distances measured from a common origin.

In this case it stands for $\frac{c-a}{d-a}\div\frac{c-b}{d-b}$.

I am not suggesting that the above notation should *supersede* the notation $(abcd)$:

If we say $(abcd)=(a'b'c'd')$, we mean that any one of the six cross-ratios typified by the one is equal to the corresponding cross-ratio typified by the other. My notation would only be used when we wish to deal with special cross-ratios out of the six possible ones, and is perhaps specially useful in the multiplication of cross-ratios of the type $\frac{a}{b}\left(\frac{c}{d}\right)\times\frac{a}{b}\left(\frac{d}{e}\right)$.

Examples of such multiplication will be found in J. J. Milne's *Cross-ratio Geometry*, p. 13, Ex. 3 ; p. 192, Art. 201 ; p. 196, Art. 207 ; p. 230, Art. 249; and p. 231, Ex. 1. And there are other parts dealing with reciprocal cross-ratios where the notation would be illuminating.

I should be glad if some readers of the *Gazette* would express their opinion of the value of the above notations, and also as to whether they are new.

ALFRED LODGE.

QUERIES.

(76) Why is it that English writers (Hymers, Salmon, Carr, Frost, C. Smith, Bell) and the American historian Cajori, invariably refer to *Meunier's* Theorem instead of *Meusnier's* Theorem? General Jean Baptiste Marie Charles Meusnier (1754-1793) first announced the theorem concerning the curvature of oblique sections of surfaces in 1776, although it was not published till 1785. From Chasles' sketch in his *Rapport sur les Progrès de la Géométrie* down to Darboux's "Historical Notice" delivered before the Academy of Sciences at Paris in 1909, no suggestion is given that Meusnier ever spelled his name in any other way. R. C. ARCHIBALD.

(77) Since the second edition (1882) of *Exercices de Géométrie*, F. G. M. has given references to "Key to Exercises on Euclid's Elements of Geometry, by William Collins." Riccardi, Poggendorff and the British Museum catalogue have no record of such a work. When, where and by whom was it published? Is it a Key to an edition of Euclid by Collins? If so, similar details concerning this edition are desired. R. C. ARCHIBALD.

(78) According to my copies of the *Diary Companion* and of Leybourn's *Mathematical Repository*, old series, there was published in London, in 1801: "Mathematical and Philosophical Tracts and Selections; containing several scarce and valuable papers in Geometry, Mechanics, Analytics, and Philosophy. Part I." Boards, 7s. 6d. A couple of years later, Part II. was announced as "in the press." Was the volume ever published?

A copy of Part I., lacking the original binding and cover, is in the Brown University Library. It contains: portions of three geometrical tracts by Matthew Stewart, John Playfair, W. Wallace; two complete articles on Mechanics and one incomplete article; two papers in "Analytics," and seven in "Philosophy." The title page of the volume is the title page of the first geometrical tract: "Geometrical Propositions, demonstrated after the manner of the ancients. Translated from the Latin" of Dr. S. Twenty-four pages only of this translation (to the beginning of Proposition XXIV.), are given, and for these pages only 29 of the 58 figures of Plate I., prepared to accompany the translation, are made use of. Was this translation ever completely published? In the *Dictionary of National Biography*, E. Irving Carlyle writes: "Stewart was the author of 'Propositiones geometricae more veterum demonstratae,' Edinburgh, 1763, 8vo; translated in 1801," while the British Museum catalogue records only the "imperfect" work of 24 pages. The probabilities seem to emphasise Mr. Carlyle's looseness of statement. Was Thomas Leybourn the translator and editor of the published portion of Stewart's tract? R. C. ARCHIBALD.

(79) In 1802 Thomas Leybourn, editor of *The Mathematical and Philosophical Repository and Review*, published a two shilling royal octavo pamphlet entitled "A Synopsis of Data for the Construction of Triangles, improved and very much enlarged upon the plan of Mr. Lawson." In connection with an advertisement of this work in my copy of the *Diary Companion* for 1803 occurs the statement: "Solutions are preparing on a general and extensive scale by the Editor and his Friends, and will be published afterwards." Were these solutions ever published?

R. C. ARCHIBALD.

(80) Some of the readers of the *Gazette* might like to try the following simple looking problem:

ABC is a triangle in which A is a right angle. D is a point on BC such that $B\hat{A}D=\frac{\pi}{8}$. $AD=10$, and on it E is taken such that $AE=7$. The triangle is such that $EB=EC$. Show that $BC=18{\cdot}2$ nearly. J. J. MILNE.

ANSWERS TO QUERIES.

[**71, p. 330, vol. v.**] (First Solution.)

Given $\phi(n+\frac{1}{2})+\psi(n)=A_0+A_1n+\ldots+A_r\frac{n\,.\,\overline{n-1}\ldots\overline{n-r+1}}{r\,!}+\ldots\equiv F(n)$,

where ϕ is an *odd* and ψ an *even* function, to prove

$$\psi(0)=A_0-\frac{A_1}{2}+\frac{A_2}{2^2}\ldots+(-1)^r\frac{A_r}{2^r}+\ldots.$$

This is clearly reducible to the case when

$$F(n)\equiv\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\ldots\overline{n-r+1}}{r\,!}\equiv\binom{n}{r}.$$

Now if $m=n+\frac{1}{2}$, $F(n)=F(m-\frac{1}{2})$, and if we remove the odd powers of m from this, we are left with

$$f_1(n)=\tfrac{1}{2}\{F(m-\tfrac{1}{2})+F(-m-\tfrac{1}{2})\}=\tfrac{1}{2}\{F(n)+F(-n-1)\}.$$

The method consists in removing from $F(n)$ alternately the odd powers of $n+\frac{1}{2}$ and the even powers of n, till only a term independent of n remains.

Now if
$$F(n)=\binom{n}{r},$$
$$f_1(n)=\tfrac{1}{2}\left\{\binom{n}{r}+\binom{-n-1}{r}\right\}.$$

Removing the even powers of n, we are left with

$$g_1(n)=\frac{1}{2^2}\left\{\binom{n}{r}+\binom{-n-1}{r}-\binom{-n}{r}-\binom{n-1}{r}\right\}=\frac{1}{2^2}\left\{\binom{n-1}{r-1}-\binom{-n-1}{r-1}\right\},$$

since
$$\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}.$$

Removing the odd powers of $n+\frac{1}{2}$, we are left with

$$f_2(n)=\frac{1}{2^3}\left\{\binom{n-1}{r-1}-\binom{-n-1}{r-1}+\binom{-n-2}{r-1}-\binom{n}{r-1}\right.$$
$$=\frac{1}{2^3}\left\{-\binom{n-1}{r-2}-\binom{-n-2}{r-2}\right\}.$$

Similarly,
$$f_3(n)=\frac{1}{2^5}\left\{\binom{n-2}{r-4}+\binom{-n-3}{r-4}\right\},$$

....

$$f_{q+1}(n)=\frac{(-1)q}{2^{2q+1}}\left\{\binom{n-q}{r-2q}+\binom{-n-q-1}{r-2q}\right\}.$$

If r is odd and $=2p+1$, the last term will be

$$\frac{(-1)^p}{2^{2p+1}}\left\{\binom{n-p}{1}+\binom{-n-p-1}{1}\right\}=f_{p+1}.$$

If r is even and $=2p$, the last term will be

$$f_{p+1}=\frac{(-1)^p}{2^{2p+1}}\{1+1\}.$$

Now the value of $\psi(0)$ is clearly

$$f_1(0)+f_2(0)+\ldots+f_{p+1}(0),$$

which

$$=(-1)^r\left[\frac{1}{2}(0+1)-\frac{1}{2^3}\left[1+\binom{r-1}{1}\right]+\frac{1}{2^5}\left[\binom{r-3}{1}+\binom{r-2}{2}\right]\right.$$
$$\left.-\frac{1}{2^7}\left[\binom{r-4}{2}+\binom{r-3}{3}\right]+\ldots\right],$$

the last term being

$$\frac{(-1)^p}{2^{2p+1}}\left[\binom{p}{p-1}+\binom{p+1}{p}\right] \text{ if } r=2p+1, \text{and } =\frac{(-1)^p}{2^{2p+1}}\left(\binom{p-1}{p-1}+\binom{p}{p}\right) \text{ if } r=2p.$$

This may be written, in the case of $r=2p+1, -\frac{1}{2}K(p)+\frac{1}{2^3}K(p-1)$,

where $$K(p)\equiv 1-\frac{1}{2^2}\binom{2p}{1}+\frac{1}{2^4}\binom{2p-1}{2}\ldots+\frac{(-1)^p}{2^{2p}}\binom{p+1}{p},$$

and in the case of $r=2p$, in the form $\frac{1}{2}L(p)-\frac{1}{2^3}L(p-1)$,

where $$L(p)=1-\frac{1}{2^2}\binom{2p-1}{1}+\frac{1}{2^4}\binom{2p-2}{2}-\ldots+\frac{(-1)^p}{2^p}{}^pC_p.$$

But it may be proved that

$$1-\frac{1}{4}\binom{r-1}{1}+\binom{r-2}{2}\frac{1}{4^2}-\binom{r-3}{3}\frac{1}{4^3}+\ldots=\frac{r+1}{2^r}.$$

Hence $$\psi(0)=(-1)^r/2^r.$$ R. F. M.

[**74, p. 159, vol. vi.**] I have shown in my *Elementary Geometry*, p. 249, that if AB is divided at X so that $AX.AB=XB^2$ and a circle is described on XB as diameter, then the tangent AT to this circle meets the perpendicular from B to AB at a point C such that $BC=\frac{1}{2}AB$. I have found that my pupils can grasp the corresponding construction more easily than any other, viz.: Draw the perpendicular $BC=\frac{1}{2}AB$; bisect the angle ACB by the straight line CO; with O as centre draw the circle BTX touching AC in T; $AX.AB=XB^2$. This is practically the same property as that enunciated by Mr. Casey. It is obviously suggested by the construction of Euc. II. 11.

CECIL HAWKINS.

[**75, p. 159, vol. vi.**] The construction in question occurred to me about nine years ago, and I worked out some applications to electricity, optics, and continued fractions, which however I have not yet published. An electrical engineer, to whom I showed it, said he had seen the same construction published about ten years earlier in a French technical journal, probably *La Lumière Electrique*. I have the impression that I verified the reference at the time. The construction is certainly an interesting and useful one. The following construction for the resistance of a set of conductors arranged as in Wheatstone's Bridge, is based on it.

x, y are the resistances in one branch, a, b those in the other, and g the galvanometer resistance in the bridge.

Let OW be the bisector of the angle VOU.

Cut off from OW, OX, OY, OA, OB equal to x, y, a, b respectively, and draw OG $\perp^r$ to OW, to represent g.

Let the mid-points of XY, AB, AX, BY be Z, C, M, N.

Let lines $A'AA''$, etc. $\perp^r$ to OW be drawn to meet OU in A', etc., and OV in A'' etc.

Let OW cut $X''A'$, $Y''B'$, $Z''C'$, $M''N'$ in D, E, F, H.

Let K be the mid-point of DE.

Join HG and draw KL $\perp^r$ to it, and draw FP making $O\hat{F}P=45°$, to meet KL in P.

Let Q be the projection of P on OW.

Then OQ represents the required resistance. R. F. MUIRHEAD.

REVIEWS.

Projective Geometry for Use in Colleges and Schools. By Wm. P. Milne. Pp. xiii + 148. 2s. 6d. 1911. (Macmillan.)

The book consists of four chapters, the first of which explains, with very clear figures, the meaning of projection, with examples illustrating its value, and lays down the cross-ratio properties of ranges and pencils, but without proof of them, references being given to other books in which the proofs may be found. In fact, throughout the book a great number of theorems are stated with references to where proofs of them can be found. All such theorems are printed in black type. Chapter ii. deals with the conic and introduces imaginary points, and in particular the circular points at infinity, which occupy a very prominent position throughout the book. Their quasi-existence, and their properties, are very clearly explained, and a student should obtain a very good grip of them and of their use in the solution of projective problems. The latter part of the chapter is devoted to orthogonal projection and its use in proving conic properties. The third chapter is devoted to reciprocation, and will well repay the most careful reading, though it seems a pity that to complete some of the proofs reference is needed to some other book (see p. 94). The last chapter deals with general properties of conics, and will be found the hardest, as the theorems requiring reference to other books are of a harder nature. It could only be read thoroughly by a student who already had read widely.

There are a great number of valuable examples at the end of each chapter. Many of these could be solved by help of the theorems and methods explained in the book, but a good many require a great deal of knowledge otherwise acquired. They form an invaluable storehouse of problems which will test to the full the powers of the best students.

Junior Mathematics. By D. B. Mair. Pp. viii + 200. 2s. 1911. (Clarendon Press.)

The chief value of this interesting book lies in its examples. Mr. Mair is one of the pioneers who are gradually building up sets of problems connected with practical life which, it is hoped, will finally replace the artificial examples with which hitherto text-books have been filled: and consequently some of the examples, notably No. 52, p. 93, are full of suggestion and interest for much older students than those for whom the book is written. The chief business of the text is with the elementary geometry of rectilinear figures and circles, including Pythagoras, areas, mean proportionals, etc., but not going so far as angle properties of circles. Incidentally a certain amount of algebra is introduced, and a very interesting and luminous exposition of the method of square root is given, aided by a geometrical diagram. In the examples are a good number of algebraic exercises on the construction and evaluation of formulae, and a number of problems to be solved graphically.

A book of this kind can hardly be taken as the sole text-book, particularly in algebra, as it does not proceed on systematic lines, but a great deal can be done with it in the hands of a competent teacher, and it can hardly fail to stimulate the interest and intelligence of the pupils.

It is intended for higher grade schools, higher elementary schools, the central schools of the London County Council, the supplementary courses of Scottish elementary schools, and for preparatory schools up to all but the higher sets. It would also be valuable for the junior forms in public schools as a problem book. There is a useful index at the end. L. A.

A Treatise on Dynamics. By A. Gray and J. G. Gray. Pp. xvi + 626. 10s. net. (Macmillan.) 1911.

This work is sure to appeal to a very large class of students of Mechanics. In scope and method of treatment it recalls some modern German treatises, and the authors state in their preface that their aim has been similar to that of Herr Föppl in his well-known *Technische Mechanik*.

It matters not in what particular application of dynamics the reader is specially interested he will find here something of value either in the freshness of treatment of the theoretical part or in the practical illustrations to which the theory

is applied. Though practical illustrations are abundant in every chapter, Chapter VII., which occupies about a hundred pages of the book, is specially devoted to applications of dynamical principles. At the commencement of this chapter the authors state that "by practical examples drawn from ordinary experience, and from engineering and the arts generally, the relations of the different fundamental ideas, and indeed their precise significance are made clear, and by a careful study of these the student can obtain a hold of the subject which no mere study of abstract formulae can provide." The revolution which has taken place in recent years in the teaching of mechanics in schools seems now to have penetrated our universities, and the barriers which so long existed between rational mechanics and applied mechanics seem to be rapidly breaking down. If mention is made of a few of the subjects discussed at the commencement a good idea of the scope of the whole chapter will be obtained: Effect of nature of road surface and efficiency of brakes for locomotives and motor cars; the cant of the rails in taking curves; the variation in speed of a rifle bullet in air; the proper height of buffers or of a line of draught.

After a chapter of rotational motion which leads up to Euler's equations, there is a chapter devoted entirely to gyrostatic motion. The authors lay claim with some justification to an originality of treatment which simplifies some of the initial theoretical parts, and this simplification will be welcomed by many students. Among the practical applications discussed are Brennan's mono-rail, Schlick's apparatus for diminishing the rolling of ships and the gyrostatic control of torpedoes. There is no mention, however, of the Anschütz gyro-compass, an instrument which probably in the near future will replace the magnetic compass.

The book ends with a short chapter on statics, which is unsatisfactory chiefly on account of its incompleteness. The treatment is analytic and academic, except for Culmann's method of getting graphically the reactions for a loaded structure. Bow's notation is, however, not employed. The introduction of this chapter serves no useful purpose, and is to be regretted.

The work seems free from serious errors, though there are one or two misprints, viz. p. 351 and p. 426. In chapter i. pp. 17, 18 and 19, the motion of a crank, connecting rod and piston is discussed at some length. The well-known graphical construction for the speed of the cross-head is proved, and then the statement is made that when the crank is at right angles to the line of motion of the piston, the velocity of the cross-head has its maximum numerical value (p. 18, line 8). This is not the case: if *the length of the connecting rod is great compared to the length of the crank*, it is easy to show that the maximum value of the speed of the cross-head occurs when *the crank is nearly at right angles to the connecting rod*. The general case leads to a somewhat intractable equation. The error is all the more surprising, as the authors work out the acceleration of the cross-head and give a neat graphical construction for the acceleration at any phase of the motion. If this construction is carried out when the crank is at right angles to the line of motion of the piston, it will be seen that there is invariably at this point a retardation, and that consequently the maximum speed occurs before this position is reached.

The Elements of the Mechanics of Materials and of Power Transmission. By William R. King, U.S.N., retired, Principal of Baltimore Polytechnic Institute. Pp. 226. n.p. 1911. (John Wiley & Sons, New York.)

This work comprises a fairly complete course of Applied Mechanics, together with two short chapters on the Transmission of Power by Belts and the Transmission of Power by Toothed Wheels. It presupposes for the student a knowledge of general Mechanics and an acquaintance with the Calculus. In a preliminary chapter a number of standard cases of Centre of Gravity and of Moments of Inertia are worked out without much explanatory matter. The chapters on Bending Moment and Shear, Theory of Beams, Deflection of Beams, and Framed Structures are excellent. The diagrams are clear and convincing, and nearly all the examples are worked out from numerical data.

In the chapter on Framed Structures mention is made of redundant and deficient frames, but not of counterbracing in framed girders and the extent of the girders over which it is necessary, though this would come well within the scope of the work.

The introduction of the term 'vector' in the treatment of the Funicular Polygon

is most unsatisfactory and misleading. Why should the stresses in the members of the funicular polygon be vectors and not the external forces acting on the polygon? An otherwise clear and consistent account of the funicular polygon method of obtaining the resultant of a system of coplanar forces is spoiled by the gratuitous introduction of this important idea.

It is stated on p. 64 that "the neutral axis invariably passes through the centres of gravity of a section whose material has its modulus of elasticity the same in tension as in compression." This is the case when the beam is subjected to transverse loads alone, but is not true if longitudinal external forces come into play; the voussoir of an arch is a case in point.

There are one or two minor points that call for comment. The sentence "$\frac{dy}{dx}$ being small $\left(\frac{dy}{dx}\right)^2$ is infinitesimal" (p. 73) is loose and inaccurate. On p. 66 we have a somewhat startling innovation in nomenclature, "I is expressed in *biquadratic* inches."

R. M. M.

Abhandlungen über die Reform des mathematisches Unterricht in Ungarn. Pp. 160. Price 4 m. 1911. (Teubner, Leipzig.)

This work, translated into German by Messrs. Beke and Mikola, contains a valuable series of reports on specific details of mathematical education, due to the activity of a commission appointed by the Hungarian association of teachers in secondary schools. The reports were drawn up by individuals, and discussed at meetings of the association. In the light of the discussion the authors subsequently formulated certain resolutions which they conceived to represent the opinions expressed. A list appended below of the topics dealt with shows that the questions agitating English mathematical circles have also troubled our neighbours, and a perusal of the reports furnishes much valuable aid towards the formation of sound opinion. Professor Beke deals with the general case for mathematical reform, and Herr Goldziher sketches the history of the movement in different countries.

The simplification of arithmetic, the place of graphical methods, the object of teaching geometry, and the extent of school courses in geometry—the place of the slide rule in school work—the time to introduce the concept of functionality and the elements of the calculus, and the relation of mathematics to physics, are then dealt with in turn. Mathematics has in this country suffered much from having held an undisputed place in school work. It is preparing for a stout defence, a defence destined to succeed: and it is an inspiring thought that mathematical teachers of every civilized nation are facing the same problems.

C. S. J.

Shorter Geometry. By C. Godfrey and A. W. Siddons. 1s. 1912. (Cambridge University Press.)

In addition to possessing the merits which one would expect from the names of the authors, this book deserves the careful attention of teachers as being the first of its kind (as far as the reviewer knows), that is, the first Geometry suitable for schools whose arrangement is based on the Board of Education's circular on the teaching of geometry.* In other words, the basis of assumption made is widened to include the fundamental propositions on angles at a point, angles made with parallels, and congruence of triangles, these facts being brought out by exercises so that the student obtains an intuitive grasp of them. The formal proofs of them are relegated to an Appendix. After this stage the book follows the usual lines.

As is natural in the case of pioneer work, some points in the arrangement are open to criticism: perhaps the most noticeable being that references are given throughout the book to these fundamental theorems by number when the student has presumably not learnt the proofs of the theorems or their numbering, but has learnt them as 'facts' in a different order, and labelled with letters.

The book is beautifully printed, and contains one artifice in this way which will be of great use to the hurried teacher, a special mark ¶ attached to those exercises which should be worked in class with guidance from the teacher, so as to lead up to the discovery of each new theorem.

C. O. T.

* *Teaching of Geometry and Graphic Algebra in Secondary Schools*, Circular 711 (Wymans, 1d.).

A New Geometry. BAKER and BOURNE. Books I. to III. 1s. 6d. 1912. (Bell & Son.)

This book may be recommended to teachers who like a cheap book, and one which contains a great deal in a little space. The standard propositions are all here (rather more of them than some teachers would wish), with plenty of exercises, and no wasted verbiage. The general arrangement of the book is on familiar lines, but the arrangement of the early propositions is ingenious. Euc. I. 16, proved from first principles, leads to the propositions on parallels; and a proof of Euc. I. 5, by folding, enables the propositions on congruence of triangles to be grouped together. C. O. T.

Konstructionen und Approximationen. By THEODOR VAHLEN. Pp. xii + 349. 11 marks. 1911. (Teubner, Leipzig.)

This book, which is Volume 33 of Teubner's *Sammlung*, is described in the subtitle as a supplement to elementary and an introduction to higher geometry.

The book is divided into eight sections, dealing respectively with linear constructions, projective and metrical; quadratic constructions, cubic constructions; algebraic and transcendent constructions: numerical approximations, analytical approximations, approximations by geometric construction: and the transcendental nature of e and π.

In the earlier portion, several good examples of the power of projective methods are collected which are not readily accessible elsewhere. As an instance may be mentioned the theorem, proved independently by H. J. S. Smith and Kortum, that with one fixed conic given, the cubic and biquadratic constructions can be effected with ruler and compass.

The book is a rich mine of problems, and abundant references to the original sources open up inquiries in many directions. C. S. J.

The twenty-seven lines upon the cubic surface. By HENDERSON. (Cambridge Tracts No. 13.) Pp. 100. 4s. 6d. net. 1911. (Cambridge University Press.)

The central problem of this tract is a fascinating one—to construct a model of the twenty-seven lines. The writer has actually solved the problem, as he says, for each of the 21 types of unruled cubic surface, by a uniform method. An account of his results forms the longest of the seven chapters. There are thirteen plates containing perspective representations of the models. On the practical side, then, the labour and care expended by the author have been rewarded.

On the theoretical side the tract is disappointing. There is no conclusive proof that the lines even exist on the general cubic surface: the author follows Salmon in counting constants, as if porisms were unknown. No reference is made to Cremona's form of the cubic

$$x^3+y^3+z^3+u^3+v^3+w^3=0,$$

where

$$x+y+z+u+v+w=0,$$

nor to the parametric representation of a variable point on the surface. On the other hand, some tables are given at needless length; six pages (31-36) are taken up with the results of permuting 123456 in the formula

$$\begin{matrix}(35)(64)(12.34.56)\\+(34)(65)(12.35.64)\end{matrix} \quad a_3, a_6;\ b_4, b_5;\ c_{12}, c_{34}, c_{56}, c_{35}, c_{46}.$$

By way of compensation the author gives a valuable bibliography and historical summary. A. C. DIXON.

A School Calculus. By A. M. MCNEILE. Pp. xii + 376. 7s. 6d. 1912. (Murray.)

This book is intended to be a simple and intelligible presentment of the fundamental principles with a view to more or less immediate application. The treatment adopted is that of employing definite numbers instead of algebraic quantities (*i.e.* arithmetical numbers instead of literal or algebraic numbers) for preliminary explanations, "so that the real meaning of new and unfamiliar ideas, limits for example, may be more readily grasped by the beginner."

It is unfortunate, therefore, that the definition of a limit is inaccurate. According to the definition given, if ϵ is any small assigned number, no matter how small, and $|f(x)-A|$—(this notation is not used)—can be made less than ϵ by taking $|x-a|$ less than some other small number η, then $\lim_{x\to a} f(x)=A$.

The authors do not seem to be aware that the idea of a limit tacitly assumes a *sequence* of values of $f(x)$—*vide* Hobson's *Functions of a Real Variable*—in which each value is at least not greater than the preceding when $|x-a|<\eta$.

For instance, we can easily write down a function which continually approaches a value 3, say, as x approaches to within say $\epsilon=10^{-6}$ of the value 1, but increases indefinitely if we go beyond this value of ϵ. Thus find $\lim_{x\to 1}(x^3-1{\cdot}000004)/(x-1)$.

The definition given should have the words "If $f(x)$ can be made to differ from a definite number A by less than ..." altered to "If the difference between $f(x)$ and a definite number A can be made to become *and remain* less than ...". Even then, it would not be unwise to insist that in all functions for elementary consideration each member of the sequence must be *less* than the preceding.

It is unwise even to suggest to an army or engineering student that he should ever substitute the value a of x at all. Far better to tell him that the whole subject is, so far as he is concerned, approximate, the error being less than anything he likes to mention.

Another unfortunate point is the introduction of the symbols $\frac{0}{0}$, 1^∞, $\frac{\infty}{\infty}$, etc., since they are meaningless. A case arises, where the proverbial "intelligent student" would ask a question very awkward to answer. With regard to the differentiation of a constant, we find

$$\text{"Let } y=c, \text{ then } \frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{c-c}{\delta x}=0."$$

Now, if the idea of substitution has once been allowed to get into the student's head, he will say "the limiting form of $\frac{c-c}{\delta x}=\frac{0}{0}$ and is therefore meaningless. Why therefore do you say it is zero?" The answer, to his satisfaction, could not be given without starting the subject all over again, and correctly.

For the student of the class catered for, far too much haste is made for thoroughness. If his attention is confined to polynomials for most part of the time until he is thoroughly conversant with the ideas of differentiation and integration, his subsequent progress will be ten times as rapid.

It is stated that as little previous mathematical knowledge as possible is assumed, and yet "it is expected that the limits of

$$\left(1+\frac{1}{x}\right)^x, \quad \frac{a^x-1}{x}, \quad \frac{\sin x}{x}, \quad \frac{\tan x}{x}$$

are *known and understood.*" This is followed two pages later by the (inaccurate) definition of a limit mentioned above.

Art. 26—Integration by parts—would be much more intelligible if u, v were used instead of $f(x)$, $\phi(x)$.

The use of the inverse hyperbolic ratios should be more frequent: when they are used in this volume the choice has been injudicious.

Thus $\int\frac{dx}{\sqrt{a+2bx+cx^2}}$ $(ac>b^2)$ is given as $\frac{1}{\sqrt{c}}\sinh^{-1}\frac{b+cx}{\sqrt{ac-b^2}}$. Surely it is preferable to retain the argument $a+2bx+cx^2=R$, say, in the form

$$\int\frac{dx}{\sqrt{a+2bx+cx^2}}=\int\frac{dx}{\sqrt{R}}=\frac{1}{\sqrt{c}}\cosh^{-1}\sqrt{\frac{cR}{ac-b^2}}.$$

The part which deals with applications is very well done, the figures and explanatory diagrams being unusually well and clearly drawn. There are 200 well chosen miscellaneous examples at the end of the book, some of which at anyrate will "make the beggar think."

There is nothing to grumble at in the get-up of the book, the type being clear and readable, but the weight and price of the volume are both on the heavy side for a beginner's book.

Exercises in Woodwork. F. E. DRURY. Pp. xi+215. 2s. 6d. 1912. (G. Bell & Sons.)

This book is a progressive course of exercises, designed to extend over three years, in which sketching, mechanical drawing, geometry and calculations go hand in hand. After discussing it carefully with my colleague, Mr. R. D. Whitehead, who has had over twenty-five years' experience in teaching engineering and building students, the conclusion arrived at was that the book is a distinctly original and valuable contribution towards making "practical mathematics" of *educative* utility. In one sense it is reactionary, for it corroborates the growing body of opinion that it does not matter *what* you teach the present day boy, whether it is plumbing or Greek, so long as you "make the beggar *think*": and this is the one thing necessary to save the teaching of practical mathematics from being the plague-spot in education. First, *thought*; secondly, thirdly and lastly, *accuracy*, if you will; but FIRST, *thought*. And this is just what we have in this book; the arrangement of the practical work is excellent, and the questions preceding each example being a succession of the three fundamentals in teaching—WHY? HOW? and WHAT? The scheme of procedure is, however, in the preface, happily qualified by saying that "it need not be rigid." The proof of the pudding is in the eating; and to many who know the so-called technical student and his ways, the exercises given, especially towards the end, will seem to be of the lightning order of get-wise-quickly species, the fitting requiring a finish such as a first-class cabinet-maker would not be ashamed of: this consumes the time which would be of more use devoted to educative purposes. A great deal will depend on the "individuality of the teacher"; and, still more, even for the first few chapters, on the way in which the student's first lessons in algebra have been presented. If the first steps in algebra have been taken by the modern logical route of generalized arithmetic, well and good; but if algebra has been taught according to the rule-of-thumb tenets of "practical mathematics" (as ordinarily practised) the attainment of good results will necessitate a teacher of exceptional merit.

Two chapters are added on graphs and one on percentage error which materially add to the value of the book. Viewed either from a theoretical or practical standpoint, this book is to be welcomed as a distinctly original effort that cannot fail to arouse interest. It is a great pity that it is put forth in such a modest fashion. Doubled in size, and issued in two parts, elementary and advanced, it would make a bold bid for universal adoption.

Plane and Spherical Trigonometry. J. G. HUN and C. R. MACINNES. Pp. 205. 6s. net. 1911. (Macmillan.)

It is almost sufficient to state of this book of 205 pages, covering the "Elements of Plane and Spherical Trigonometry," that 100 pages are 'Tables and how to use them.' All that is considered necessary for "Trigonometry up to the solution of Triangles," as the phrase goes, is condensed into sixty-two pages. It is essentially a college hand-book, but totally unsuitable for beginners in schools. Two remarks in the preface obtrude themselves. "The fact that the trigonometric functions are ratios of line segments is emphasized, and their representation by means of lengths of lines is used as little as possible." What is the length of a line? Is it not a ratio of one line to another, the length of the second line being called the unit of length?

Again, "Certain of the proofs of theorems are shorter than in many text-books, and, it is hoped, thereby made more clear. Notably the proofs of the formulae for sin $(A \pm B)$ and cos $(A \pm B)$." We recently reviewed a trigonometry in which these proofs were half as short, and twice as clear and valid for A, B acute or obtuse so long as $A+B$ is less than two right angles. J. M. CHILD.

Über die Theorie benachbarter Geraden und einen verallgemeinerten Krümmungsbegriff. By W. FRANZ MEYER. Pp. 152. Price 8 marks. 8vo. 1911. (Teubner.)

The generalized curvature bears the same relation to any straight line g through a point P of a curve, as curvature in the usual sense bears to the tangent t; this substitution of g for t is the main idea on which the book rests. First, there is constructed the "begleitende Dreikant" of g, corresponding to the

canonical trihedron whose axes are the tangent, principal normal and binormal to the curve. The derivates, with regard to the arc, of the direction cosines of the trihedron accompanying g are related to the cosines themselves by generalized Frenet formulae, whose form is the same as in the canonical case. These equations are interpreted geometrically by means of three spherical curves traced out by radii parallel to the moving axes.

The greater part of the book is devoted to the expansions, in powers of the increment ds of arc, of (i) the shortest distances between adjacent positions of the moving axes, (ii) the distances from P of the feet of these shortest distances, and (iii) the projection of ds upon the axes of the trihedron; and attention is paid to the special cases in which the expansions begin with higher powers of ds than in the general case. It is proved, among other things, that the shortest distance between adjacent tangents varies as ds^3, and cannot vary as a higher power of ds unless it vanishes absolutely. The formulae are re-stated for the canonical trihedron, and also for the trihedron associated with a curve on a surface, where g is the normal to the surface.

The first appendix extends many of the results to space of n dimensions, and the second investigates curves on a surface by means of in- and co-variants.

The desired results are pursued by a systematic method of successive approximations, which involves a great mass of algebra, in contrast to the neat formulae which result. There is hardly a symbol without a suffix, some of them alarmingly complicated, and praise is due to the clear printing of Messrs. Teubner, to whom the volume is dedicated. H. P. H.

Lectures on Fundamental Concepts of Algebra and Geometry. By J. W. YOUNG: prepared for publication with the co-operation of W. W. DENTON; with a note on **The Growth of Algebraic Symbolism.** By U. G. MITCHELL. Pp. vii+247. 7s. net. 1911. (Macmillan & Co.)

The movement for reform in the methods of mathematical teaching may be directed with skill and energy, but it must fail, at any rate for the time, to attain a full measure of success, unless the whole teaching body becomes informed by the new spirit of mathematical thought. Our continental brethren have of late years had the opportunity of breathing an atmosphere from which a very large proportion of our secondary school mathematical teachers is unfortunately excluded. We are running the danger of drawing upon a bank cheques which may be returned to us endorsed with the inscription "no account." As a single instance, take the recent suggestion by Mr. G. St. L. Carson that it is wise at some stage of the school career to widen the boy's horizon by giving him an outline of the history and concepts of non-Euclidean geometries. Yet there must be a considerable percentage among the teachers in secondary schools who would be hard put to it even to name the sources from which they would obtain the material required for a conversation or a lesson upon such a subject. It cannot be other than wholesome for the inquiring mind to be brought to the frontier of regions which still abide our quest, and to learn that there are still realms left to conquer, that there is no such thing as finality in mathematical thought. Lectures such as those in the volume before us are therefore heartily to be welcomed in so far as they help to fill a gap, the existence of which in English mathematical literature is much to be deplored.

References given to articles in these or those sets of proceedings, or in volumes in foreign languages, are idle in far too large a number of cases, if we remember both the miserable stipends on which so many have to eke out a livelihood, and the difficulty of access to such proceedings, etc., which is experienced even by men engaged in original work of the most advanced type. We think that the author has been well advised in limiting the size of the canvas on which his pictures of the fundamental concepts are sketched in. In opening up points of view that are almost entirely novel to the reader the *ébauche* is more effective than the more ambitious effort. Here is ample material to set the teacher's mind running in new grooves, to give to his general outlook a trend which cannot fail in turn to have its effect upon the minds of those for whose training he is responsible. Professor Young's first three lectures contain a criticism of Euclid's Elements, an introduction to the non-Euclidean world, and the history of the parallel postulate. It is well brought home to the reader that the strictly logical science must have as its starting point a set of terms that are undefined and of propositions that are

unproved. The ideas of class and "belonging to a class" are then examined, and from two undefined terms and seven assumptions instances of formal reasoning are exhibited. The consistency, independence, and categoricalness of such sets are then discussed; and there emerges the idea that under any branch of mathematics there lies an abstract science the study of which is essential to a clear grasp of the logical foundations. Next we have the treatment of classes, correspondence, and number followed by that of order and discrete sequences. Cardinal numbers with the linear continuum occupy a short discourse, and by the time the twelfth lecture is over the teacher will know something of numbers, both negative and complex. Turning then to geometry, we have a detailed discussion of Hilbert's axioms, and an abstract of Pieri's kinematical theory; the possible geometries in three and more than three dimensional space are also briefly handled. Lecture 18 deals with the abstract equivalence of Algebra and Geometry, and the notions of "variable," "function," and "limit" are next set forth. Finally we are led to the borders along which metaphysics and mathematics must seek each other's aid. From time to time the lecturer halts to draw the attention of the teacher to the special lessons for his work that are afforded by the methods under discussion. But for this the book might be intended for any individual of ordinary intelligence and capacity.

On p. 194 it is stated that Descartes, in 1637, was the first to use the word "function," and that he employed it simply to denote an integral power of a variable. According to the *Encyclopédie des Sciences Mathématiques*, this statement, originally due to d'Alembert, and repeated, probably without examination, by Lagrange, Lacroix and Cournot, is incorrect. It is suggested that as in the sixteenth century the words "dignité" and "fonction" were synonymous—dignitas was used, for instance, by Tartaglia in 1556 for the product of more than two equal factors—d'Alembert may not have troubled himself with further enquiries, and translated "dignitas" by "fonction." Or, he may have merely meant to convey that the only functions to which the attention of analysts at first was drawn were the powers of the variable. Newton, in the "Methodus Fluxionum," 1671, used the phrase "relata quantitas" almost in the sense of "function." It was not until 1694 that Leibniz, in a letter to Jean Bernoulli, used in the modern sense his phrase "quantitas formata," followed by an indication of the independent "variable." (*Encyc. des Sc. Mathques.* Tome ii. vol. i. Fasc. 1.)

The remark that Diophantus . . . . "seems to have been actually the first to have made use of rational numbers" might have been more happily phrased. On p. 237 the statement that Girard "introduced parentheses and the signs $\sqrt[2]{}$, $\sqrt[3]{}$ for square and cube root" may give a wrong impression. While it is true that he "proposed" the notation, he generally conformed to the older forms, writing $(\frac{1}{2})8$, $(\frac{1}{3})9$ for $\sqrt{8}$, $\sqrt[3]{9}$, and even the Cossic signs for $\sqrt[3]{3}$ and $\sqrt[4]{4}$. The modern notation is systematically employed, according to the *Encyclopédie*, from the time of Rolle's *Traité d'Algèbre*, 1690. Round brackets were used for the grouping of terms by Tartaglia 73 years before Girard, and by Clavius in 1608. "Toutefois A. Girard réalise un petit progrès dans l'emploi des parenthèses en en faisant usage, quoique très rarement comme signe de multiplication" (G. Eneström.)

In conclusion, we congratulate Prof. Young on his attractive volume. We feel sure that no reader, teacher or other, provided that he possesses the very modest mathematical equipment represented by a modicum of elementary algebra and geometry, can fail to rise from the study of this book without feeling that a new world has dawned upon him, and that his mental attitude towards "the queen of the sciences" is informed by a new spirit.

Graph and Conic Templates.

We are glad to draw the attention of teachers and others to the above templates from the design of Mr. J. T. Dufton, which are to be obtained through Messrs. Macmillan. The Graph Template (6d. in celluloid and 3d. in nickel) is for use with squared paper, English and metric scales. Rectangular hyperbolas and parabolas of large size can be readily drawn, real and imaginary roots of quadratics can be found, and with the aid of standard curves many higher equations can be solved. It is also useful in drawing curves not only in mathe-

matics but in physics, chemistry and in mechanics. The Conic Template gives a parabola, one inch unit, an hyperbola with asymptotes at 60°, and an ellipse, the directrices coinciding with the latera recta of the parabola and hyperbola. They are 8d. and 4d. in celluloid and nickel respectively. There is also a confocal stencil giving parabolas to scales 2 cms. and $\frac{1}{2}$ in. confocal ellipses, $1\frac{1}{4}$ in. distance between foci; hyperbolas with axes, directrices and latera recta marked, the asymptotes making angles of 80° and 100°. The two ellipses and these hyperbolas give confocal conics. There is also a rectangular hyperbola $xy=1$, to scale $\frac{1}{2}$ in. They are not so large as the curves received some time ago from Messrs. Brooks & Co., in which the scale for parabola and hyperbola was 1 in. unit, the ellipse with major axis 3 in. and minor axis 2 in. They also included a cycloid, roulette of circle 2 in. diameter, and the curve $y=x$, unit one inch. If we remember rightly they could be obtained at 1s. each. The advantage of an accurately constructed figure to the student of geometrical conics is obvious enough, and those who have these useful aids will find them ot material benefit.

Tables of Logarithms and Anti-Logarithms to Five Places. By E. Erskine Scott. Student's edition. Pp. 384. 5s. net. 1912. (C. & E. Leyton, 56 Farringdon Street.)

Table of Logarithms and Anti-Logarithms (four figures), 1 to 10,000. Arranged by Major-Gen. J. C. Hannyngton. Pp. 41. 1s. 6d. net. 1912. (C. & E. Leyton.)

The Erskine Scott Tables have long been held in high repute by those engaged in actuarial and similar work. The publishers have now issued a Student's Edition, using the stereotyped plates employed in printing the revised edition of 1892, and omitting from the larger volume whatever does not seem necessary for the practical computer. The logarithms of all natural numbers from 1 to 99,999 form the first table, and the second comprises the natural numbers corresponding to all logarithms from ·00001 to ·99999, the latter being printed on green paper. The type is beautifully clear. General Hannyngton's four-figure tables of logs and anti-logs is nicely printed, and is prefaced by a short explanation of the use of logarithms. Considering the thick board binding, the quality of the paper, and the size of the type, it cannot be said to be dear at 1s. 6d. net.

Annuaire pour l'An 1912. 1fr. 50c. net. Pp. 692+124. 1912. (Gauthier-Villars.)

This handy and compact volume, a phenomenon of cheapness, makes its appearance with unfailing regularity. The first 350 pages are this year devoted to astronomy. We may note, on the authority of Prof. E. W. Brown, that the table of elements of the asteroids is not quite reliable. The second part consists of physical and chemical tables. Finally we have two articles: The mean temperature of the various districts in France, by M. Bigourdan (the "mean" being practically the arithmetical mean of twenty-four observations taken from hour to hour, day and night), and an introduction to the Method of Mean Squares by M. Hatt.

CORRESPONDENCE.

Navy Yard,
Portsmouth, N.H., *Jan.* 27, 1912.

To Editor, *The Mathematical Gazette.*

Dear Sir,—I am glad to have your note* of Nov. 28, 1911, in regard to the question of Mathematical and Engineering standpoints of Descriptive Geometry, and only regret that press of work has delayed my answer, as this is a subject that I have thought much about and have discussed with many people. In fact, the text book I have written was the result of the constant dissatisfaction expressed by practical men with methods of instruction in Descriptive Geometry.

Before my book was written the prevailing text book on the subject was

* This was to elicit a reply to the reviewer's query on p. 165. [W. J. G.]

written by Church. In fact, it still is the one principally used in this country as it takes time to get a new method known and used. I am glad to know and to inform you, however, that my book has been adopted by at least three of our large universities.

The study of Descriptive, as ordinarily conducted, is a very interesting mental drill, and the imagination is trained and developed. It is considered by many instructors as the one mathematical study that must be given the student at whatever cost.

My interest in this was due to duty at the U.S. Naval Academy where I was in charge of Mechanical Drawing. Incidentally I wrote a text book on that subject which has been and is still used there and in other schools with great success for the last twelve years. With the more advanced students we took up the making and reading of complicated practical drawings. After doing this work for some days, one of the Midshipmen remarked in a surprised way, "Why, this is Descriptive Geometry!" This branch he had studied two years before in the Mathematical Department. After he recognized the kind of work he advanced more rapidly, but kept complaining that he had studied Descriptive in the First Quadrant, and that it was difficult to follow the practical Drawings made in the Third Quadrant. This started my ideas along that line, and I realized for the first time that my own study of Descriptive, which I loved, was something that bothered me in reading practical drawings. It still does in spite of the work put on the book and the use of the cages.

It will be noted that the Practical Drawings in France are in the same quadrant as in the Descriptive Geometry, but that in both England and the United States the practical drawings are all in the third quadrant. So, the methods usually adopted for the study as a mathematical proposition are right in the line of progress for the practical men in France, but this study in your country and mine is a detached idea that is quite interesting but does not lead directly to practical utility. With us everything tends to study that may be used immediately in practice, and no time can be used for beautiful theoretical propositions that cannot at once be used in practice the day after graduation. I am not saying that this is the best way for general development, but it is the way of to-day.

So, since the mental drill is just as good in the third quadrant as in any other, why not use that one and prepare the student for the business of life? Also, he can do clear thinking in this way, whereas otherwise he must try to forget what he has so laboriously learned.

A few years later than the time I spoke of above I was the head of the Department of Marine Engineering and Naval Construction at the U.S. Naval Academy, and the subject of Descriptive Geometry was turned over to my Department—to my delight. I taught it as a part of Mechanical Drawing, running in the Descriptive as fitted with the progress in Drawing. I immediately started the ideas now in book form, and we were all at once impressed with the fact that the Midshipmen were actually understanding and liking Descriptive. Before that, as taught in Mathematics, it was considered a grind and a few only would understand it and would work out the problems for the others to copy. Also, the Head of the Department of Mathematics told me that he never expected the Midshipmen to get much out of Descriptive.

Then, the Midshipmen drifted without realizing it into practical drafting, and the course is so successful that no more time is required for the two branches of Mechanical Drawing and Descriptive Geometry than used to be required for Mechanical Drawing alone.

Speaking of the study of this subject as a drill, my friend and classmate, Prof. Spangler of the University of Pennsylvania, Dean of Engineering, teaches Descriptive in all four quadrants, so that his graduates are ready to see any drawing, no matter how drawn. This is, of course, ideal, but few of us have the time for the students to do this.

Pardon my long dissertation, but you must remember that you brought it on yourself by requesting an enthusiast to give his ideas.

Yours very sincerely,

F. W. Bartlett,
Captain, U.S. Navy.

NOTICE.

The report of the "National Committee of Fifteen on Geometry" Syllabus, which has been under consideration for nearly three years, has finally been published in a pamphlet of 80 pages, and is ready for distribution to teachers of geometry, and all others interested. This report was prepared under the joint auspices of the American Federation of Teachers of the Mathematical and Natural Sciences and the National Education Association. It includes a historical introduction and sections on axioms and definitions, on exercises and problems, and the syllabus itself, including both plane and solid geometry. It is hoped by the committee that this report may be of great service to all teachers of geometry, and to this end that it may have a wide distribution among all interested. Copies may be secured gratis upon application to the Commissioner of Education, Department of the Interior, Washington, D.C.

THE PILLORY.

The following question is taken from The Board of Education Examination in Stage III. Mathematics.

"Extract the square root of $7+3\sqrt{3}+4\sqrt{(9\sqrt{3}-15)}$."

The answer required is presumably $2\sqrt{3}+\sqrt{(3\sqrt{3}-5)}$.

Could not the Mathematical Association send out missionaries to convert the Board of Education examiners to more modern ideals? I am sure that the artificial character of the Board's mathematical papers is responsible for much of the distrust with which the subject is regarded by many evening students.

H. P.

THE LIBRARY

The Librarian acknowledges with thanks the receipt of three books, presented to the Library by Mr. W. Gallatly.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue will be issued to members in due course, containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 copy „ No. 3.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important).
1 copy „ Nos. 1, 2.

ERRATA.

Vol. VI. p. 101, l. 14, delete first sentence.
p. 221, note 364, l. 3; for *ICU* read *AIU*.
l. 5; for *DI* read *DL*.
for *I* read *L*.
figure; for *C* (on *AB*) read *C'*.
p. 222 l. 1; for *UC* read *UC'*.
p. 224, note 368, l. 4; for bc^2 read 60°.
figure; for *A* (mid *BC*) read *A'*.
l. 2, below figure; for *GCB* read *CDG*.
Vol. V. p. 330. *Query* 71, to line 4 add $=\psi(0)$.

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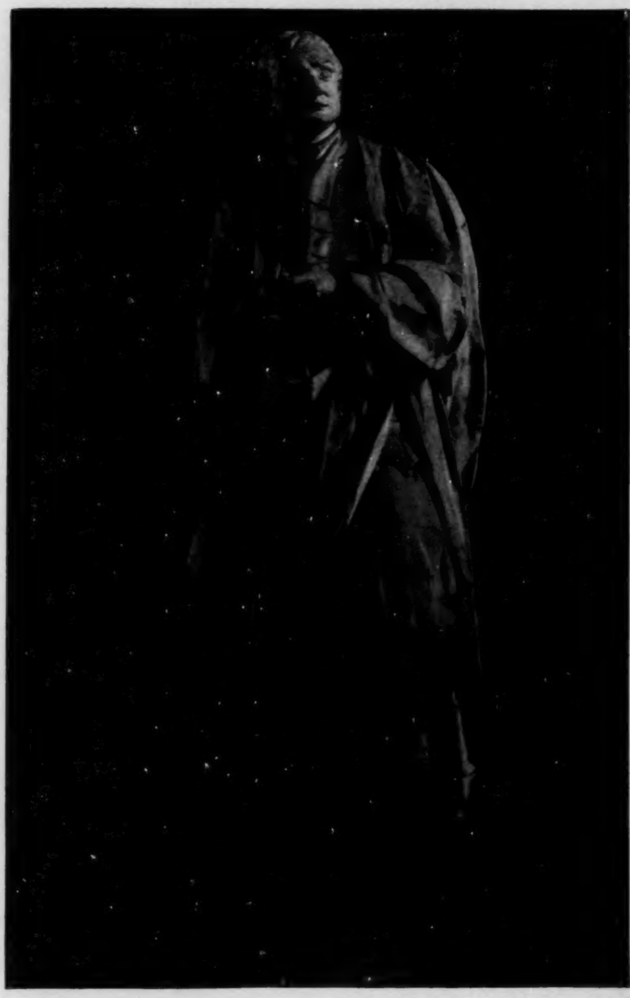

"... Newton with his prism and silent face,
The marble index of a mind for ever
Voyaging through strange seas of Thought alone."
WORDSWORTH, *The Prelude*, III.

Facing p. 311.